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Duality for Anticonvex Programs

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Abstract. Calling anticonvex a program which is either a maximization of a convex function on a convex set or a minimization of a convex function on the set of points outside a convex subset, we introduce several dual problems related to each of these problems. We give conditions ensuring there is no duality gap. We show how solutions to the dual problems can serve to locate solutions of the primal problem.

Key words: Conjugacy, Duality, Even convexity, Nonconvex duality, Polar set, Quasiconvex function, Radiant set, Reverse convex program, Shady set

1. Introduction

We consider the two problems

$$(\mathcal{M}) \quad \text{maximize } f(x) \ : \ x \in F$$
$$(\mathcal{R}) \quad \text{minimize } f(x) \ : \ x \in X \setminus C,$$

with a special emphasis to the cases the objective function f is convex and the feasible set F and the excluded set C are convex. In such cases we say that these problems are anticonvex. These two cases are difficult to deal with, either from a theoretical viewpoint or from a numerical viewpoint, although some convexity properties are present. Thus they have been studied actively in the last few years (see for instance [11, 13–15, 37–42, 45–50]). Here we study the links between these two problems and we establish various duality relationships with related problems. Our main tool is nonconvex duality.

Several duality schemes have been used for nonconvex problems, especially in the quasiconvex case (see the recent surveys [22, 27] and their references). Such problems are important for mathematical economics ([6–8, 17, 18] and some structure problems [12]). Among the proposed duality schemes, the ones devised by Atteia-Elqortobi [1, 24], Thach [40–42] and their variants by Rubinov and Simsek [30], [31], Rubinov and Glover [29] are particularly attractive because they do not require the introduction of an extra parameter as it is usually the case for duality for generalized convex problems. They are adapted to the class of radiant functions, a function f being called radiant if its nonempty sublevel sets are closed, convex and contain 0. Such a function obviously attains its minimum at 0 (in [40–42], it is even required that $f(0) = \inf f(X\backslash\{0\})$, but this stronger assumption is not necessary). In [40–42], the biconjugate of such a function f does not coincide with f unless f is upper semicontinuous (u.s.c.), a restrictive assumption we wish to avoid, indicator functions of closed subsets being of great use in optimization theory (such functions are l.s.c., not u.s.c.).

In order to do so, we modify the definition of the conjugate of f, using open half spaces as in [1] instead of closed half spaces and changing the value of the conjugate at 0. It appears that such a slight change has appealing consequences in terms of sublevel sets and, clearly, sublevel sets are important for quasiconvex functions. Moreover, the conjugacy and the duality relationships we obtain do not require some extra assumptions needed in [40]. Our work is also motivated by the fact that it seems that additional hypothesis should be given in order to make valid some statements of [40–42]. In particular, for a convex subset D of X, the equivalence

$$x \in \mathrm{int}D \Leftrightarrow \sup\{\langle x, y\rangle : y \in D^o\} < 1$$

used in [41] for a convex set D containing 0 holds when the polar set D^o of D is compact but may fail otherwise (take $D := \{(r, s) \in \mathbb{R}^2 \ : \ 2s \geqslant r^2\}$, $x = (0, 0)$). Here we make a systematic use of the *strict polar* set

$$D^\wedge := \{y \in X^* : \ \forall x \in D \ \ \langle x, y\rangle < 1\}$$

along with the usual polar set and we rely on the study of the conjugacies made in [21]. These conjugacies have the advantage of entering into the general framework of [19], [3] and others (see the monographs [20, 36] and their references in this respect) for which the conjugate of f is given by

$$f^c(y) := -\inf_{x\in X}(f(x) - c(x, y)), \tag{1.1}$$

where $c : X \times Y \to \overline{\mathbb{R}}$ is a coupling function and where the usual convention $(+\infty) - (+\infty) = +\infty$ is adopted. In such a way, known results or tools (such as perturbational duality, subdifferentials) can be used easily, as in [16, 23–25]. This observation could not be done in [40–42] because the conjugate f^H of a function used there is not of the Fenchel-Moreau type. Moreover, the dual problems we introduce seem to be more natural than the ones considered in [40–42] and we avoid the additional assumptions contained in Definitions 5.1 and 5.2 of [40]. In particular, they are easily related to the original problem and each of their feasible solutions gives a means to measure the accuracy of an approximate solution to the primal problem.

In the following section we recall the main features of the conjugacies we will use. We study in Section 3 a reverse convex program. Section 4 is devoted to the maximization of a function on a convex set. In Section 5 we gather the results of the preceding two sections and we combine them. Some applications are treated in Section 6; we refer to [26, 37–39, 40–43, 45–50] for algorithms and more substantial developments in this area.

2. Conjugacy for radiant functions

In the sequel X and Y are normed vector spaces in duality (they could be locally convex spaces in fact) and $\langle\cdot,\cdot\rangle$ is the usual coupling function. Thus the situation is entirely symmetric: X (resp. Y) is the dual of Y (resp. X) for the weak topology; but X (resp. Y) is not necessarily the dual of Y (resp. X) for the strong topology. However, each space can always be considered as a vector subspace of the dual space of the other one. When we consider closed subsets in X or Y, it is always with respect to a topology compatible with this duality. When X is a dual space and one takes for Y a predual, one has to be careful with such an assumption which does not coincide with the corresponding one in which Y is the strong dual X^*. Given $y \in Y\backslash\{0\}$ we consider the two half-spaces

$$G(y) :=\{x \in X : \langle x, y\rangle < 1\},$$
$$H(y) :=\{x \in X : \langle x, y\rangle \leqslant 1\}.$$

Let us say that a closed convex (resp. evenly convex) subset C of X is *radiant* (resp. *evenly radiant*) if it is an intersection $C = \bigcap_{y\in Z} H(y)$ (resp. $C = \bigcap_{y\in Z} G(y)$) of a family of such half-spaces (here Z is some subset of $Y\backslash\{0\}$). It follows from the bipolar theorem that a subset C of X is radiant and closed iff it is a closed convex subset containing 0. It can be shown (see [21]) that a subset C of X is evenly radiant iff it is an evenly convex subset containing 0. Recall that C is said to be *evenly convex* if it is an intersection of open half-spaces.

Setting

$$c^o(x, y) := -\iota_{X\backslash H(y)}(x),$$
$$c^\wedge(x, y) := -\iota_{X\backslash G(y)}(x),$$

where ι_S is the indicator function of a subset S of X, given by $\iota_S(x) := 0$ if $x \in S$, $+\infty$ otherwise, we get two coupling functions c^o and $c^\wedge$ which allow us to consider two kinds of conjugate functions according to the general Fenchel-Moreau scheme recalled in (1.1)

$$f^o(y) := \sup\{-f(x) :\ x \in X,\ \langle x, y\rangle > 1\},$$
$$f^\wedge(y) := \sup\{-f(x) :\ x \in X,\ \langle x, y\rangle \geqslant 1\}$$

with the usual convention $\sup\emptyset = -\infty$ (or α if the functions are considered as functions with values in an interval $[\alpha, \omega]$). The properties of these conjugacies and of related ones have been studied independently in [21], [36]; see also the bibliographies of these references, in particular [1], [9], [10] for the first conjugacy.

Let us recall their main features. The following characterization of sublevel sets is a particular case of a general rule about sublevel sets of conjugate functions with respect to a polarity ([52] Théorème I.1.6). We provide a direct proof for completeness.

PROPOSITION 2.1. *For any function $f : X \to \overline{\mathbb{R}}$, the conjugates f^o, $f^\wedge$ are quasiconvex and in fact are l.s.c., radiant and evenly radiant respectively, in the sense that for any $r \in \mathbb{R}$ their sublevel sets*

$$[f^o \leqslant r] = [f < -r]^o \qquad [f^\wedge \leqslant r] = [f < -r]^\wedge$$

are closed, radiant and evenly convex, radiant respectively.

Proof. The result follows from the equivalences

$$\begin{aligned} y \in [f^o \leqslant r] &\Leftrightarrow (x \in X, \langle x, y\rangle > 1 \Rightarrow f(x) \geqslant -r) \\ &\Leftrightarrow (x \in X,\ f(x) < -r \Rightarrow \langle x, y\rangle \leqslant 1) \\ &\Leftrightarrow y \in [f < -r]^o \end{aligned}$$

and the analogous ones with $f^\wedge$ and strict polar sets. □

COROLLARY 2.2. *The biconjugate $f^{oo} := (f^o)^o$ of any function f is such that for any real number r*

$$[f^{oo} \leqslant r] = \bigcap_{s>r} [f < s]^{oo} = \bigcap_{s>r} [f \leqslant s]^{oo}.$$

Similarly, the biconjugate $f^{\wedge\wedge} := (f^\wedge)^\wedge$ of any function f is such that for any real number r

$$[f^{\wedge\wedge} \leqslant r] = \bigcap_{s>r} [f < s]^{\wedge\wedge} = \bigcap_{s>r} [f \leqslant s]^{\wedge\wedge}.$$

These formulae characterize these biconjugates.

Proof. One has

$$\begin{aligned} [f^{oo} \leqslant r] &= [f^o < -r]^o \\ &= (\bigcup_{s>r} [f^o \leqslant -s])^o \\ &= \bigcap_{s>r} [f^o \leqslant -s]^o \\ &= \bigcap_{s>r} [f < s]^{oo} \end{aligned}$$

and similar relations for the strict biconjugate. As for $r < s < t$ one has $[f < s]^{oo} \subset [f \leqslant s]^{oo} \subset [f < t]^{oo}$, the second equalities hold. □

COROLLARY 2.3. *For any function f, its biconjugate f^{oo} (resp. $f^{\wedge\wedge}$) is the greatest l.s.c. quasiconvex (resp. evenly quasiconvex) function taking the value $-\infty$ at* 0 *majorized by f.*

Proof. Clearly, f^{oo} is a l.s.c., quasiconvex function taking the value $-\infty$ at 0 and $f \geqslant f^{oo}$. If g is a l.s.c., quasiconvex function taking the value $-\infty$ at 0 and $f \geqslant g$, then for each $r \in \mathbb{R}$ and each $s > r$ one has $[f \leqslant s] \subset [g \leqslant s] = [g \leqslant s]^{oo}$ hence

$$[f^{oo} \leqslant r] = \bigcap_{s>r}[f \leqslant s]^{oo} \subset \bigcap_{s>r}[g \leqslant s]^{oo} = \bigcap_{s>r}[g \leqslant s] = [g \leqslant r],$$

so that $g \leqslant f^{oo}$. The proof for $f^{\wedge\wedge}$ is similar. □

Note that the biconjugate f^{HH} of a function f for the conjugacy considered in [40] does not always satisfy the relation $f^{HH}(0) \leqslant f(0)$ but has the advantage of giving a more realistic value to the biconjugate at 0. The next corollary shows how one can circumvent this difficulty.

COROLLARY 2.4.

Let $f : X \to \mathbb{R}\cup\{\infty\}$ be a function such that $f(0) = \inf f(X)$. Then $f(x) = f^{oo}(x)$ (resp. $f(x) = f^{\wedge\wedge}(x)$) for each $x \in X\backslash\{0\}$ iff f is l.s.c. (resp. evenly quasiconvex) on $X\backslash\{0\}$ and quasiconvex.

Proof. The condition is necessary by the preceding corollary and the assumption $f(0) = \inf f(X)$. In order to see that it is sufficient, we introduce the function g which coincides with f on $X\backslash\{0\}$ and takes the value $-\infty$ at 0. Then g is quasiconvex and l.s.c. (resp. evenly quasiconvex) and the preceding corollary shows that $g = f^{oo}$ (resp. $g = f^{\wedge\wedge}$). □

Obviously one has $f^{\wedge} \geqslant f^{o}$. It is convenient to introduce a terminology for the cases in which equality holds. The one we coin acknowledges the efforts made in [40–42] to deal with cases in which this equality is useful.

DEFINITION 1. A function f is said to be a Thach function if $f^{\wedge} = f^{o}$.

A criteria for such a property is as follows; it incorporates the case the function is shady, i.e. is nonincreasing along rays emanating from 0.

LEMMA 2.5. *Suppose that f is quasi-shady in the sense that for each $x \in X\backslash\{0\}$ and each $s > f(x)$ there exists $t > 1$ such that $f(tx) < s$. Then f is a Thach function. In particular, any function such that for each $x \in X\backslash\{0\}$ the radial function $f_x : r \mapsto f(rx)$ is nonincreasing or u.s.c. is a Thach function.*

Proof. Given $y \in Y\backslash\{0\}$ and $s < f^{\wedge}(y)$ we can find x such that $\langle x, y\rangle \geqslant 1$ and $-f(x) > s$. As f is quasi-shady, there exists $t > 1$ such that $-f(tx) > s$. As $\langle tx, y\rangle > 1$ we get $f^{o}(y) > s$. Thus $f^{o}(y) = f^{\wedge}(y)$; as this relation obviously holds for $y = 0$, the result is proved. □

EXAMPLE 2.1. Let X be a real Hilbert space and let $A : X \to X$ be a continuous injective semi-definite positive, linear symmetric operator. Let f be given by $f(x) = \frac{1}{2}(Ax \mid x)$, where $(\cdot \mid \cdot)$ denotes the scalar product. The preceding lemma shows that f is a Thach function. Denoting by f^* the Fenchel conjugate of f, given by

$$f^*(y) = \frac{1}{2}\left\|A^{-\frac{1}{2}}y\right\| \quad \text{for } y \in R(A^{\frac{1}{2}}),\ +\infty \ \text{ for } y \in X \backslash R(A^{\frac{1}{2}}),$$

where $R(A^{\frac{1}{2}})$ is the range of the square root $A^{\frac{1}{2}}$ of A and $A^{-\frac{1}{2}}$ is the inverse of $A^{\frac{1}{2}}$ ([5] Theorem I 34), and using [21] Proposition 4.1, we get

$$\begin{aligned} f^o(y) =& f^\wedge(y) = \inf_{r \geqslant 0}(f^*(ry) - r) \\ =& -\frac{1}{2}\left\|A^{-\frac{1}{2}}y\right\|^{-2} \quad \text{for } y \in R(A^{\frac{1}{2}}),\ +\infty \ \text{ for } y \in X \backslash R(A^{\frac{1}{2}}). \end{aligned}$$

□

The two conjugates we consider have distinct features; while f^o is (weakly*) l.s.c., it is possible to ensure that $f^\wedge$ is (strongly) upper semicontinuous (u.s.c.). In the following criterion, which is similar to [40] Theorem 3.2, we say that f is *quasi-coercive* if for each $r < \sup f(X)$ the sublevel set $[f \leqslant r]$ is bounded. Equivalently, f attains its supremum at infinity in the sense of [40], i.e. for any sequence (x_n) in X such that $(\|x_n\|) \to \infty$ one has $(f(x_n)) \to \sup f(X)$. This property is also equivalent to the fact that the sublevel sets of f are either bounded or the whole space.

LEMMA 2.6. *Let X be the dual of the n.v.s. Y. If f is weakly* l.s.c. and quasi-coercive on X, then $f^\wedge$ is (strongly) u.s.c. on Y and $f^\wedge(0) = \inf f^\wedge(Y)$. If g is (strongly) u.s.c. on Y and if $g(0) = \inf g(Y)$, then $g^\wedge$ and g^o are quasi-coercive on X.*

Proof. Let (y_n) be a converging sequence in Y with limit y. Suppose that $f^\wedge(y) < \limsup_n f^\wedge(y_n)$. Then, there exist a real number $q > f^\wedge(y)$, an infinite subset N of the set of integers such that $f^\wedge(y_n) > q$ for each $n \in N$. By definition of $f^\wedge$ one can find $x_n \in X$ such that $\langle x_n, y_n\rangle \geqslant 1$ and $f(x_n) < -q < \sup f(X)$. Then (x_n) is bounded, hence has a weak* cluster point x_∞, and $f(x_\infty) \leqslant \limsup_{n\in N} f(x_n) \leqslant -q$, as f is weakly* l.s.c. By continuity of the coupling function on bounded sets, we get $\langle x_\infty, y\rangle \geqslant 1$. It follows that $f^\wedge(y) \geqslant -f(x_\infty) \geqslant q$, a contradiction.

Suppose g is u.s.c. on Y. Let $r < \sup g^\wedge(X)$. There exists $w \in X$ such that $r < g^\wedge(w)$. Let $z \in Y$ be such that $r < -g(z)$, $\langle w, z\rangle \geqslant 1$. Then, if $g(0) \leqslant \inf g(Y)$, we have $g(0) \leqslant g(z) < -r$ and, as g is u.s.c. at 0, the set $[g < -r]$ is a neighborhood of 0. Therefore, $[g^\wedge \leqslant r] = [g < -r]^\wedge \subset [g < -r]^o$ is bounded: $g^\wedge$ is quasi-coercive. The proof for g^o is similar. □

When f takes its values in an interval $[\alpha, \omega]$ with $\sup f(X) = \omega$ and when we set $f^o(0) = f^\wedge(0) = -\omega$, a natural convention for $\sup\emptyset$ in the complete lattice $[-\omega, -\alpha]$, we get that $f^\wedge$ is continuous at 0, with $f^\wedge(0) = \inf f^\wedge(X)$ whenever f is quasi-coercive; on the other hand, when $g : Y \to [-\omega, -\alpha]$ is continuous at 0 and $g(0) = \inf g(Y)$, one has that g^o and $g^\wedge$ are quasi-coercive, with $\sup g^o(X) = \sup g^\wedge(X) = -g(0)$. The proofs of these assertions are identical to the ones of [40], Theorem 3.2. The preceding convention is especially attractive when $\alpha = 0$, $\omega = +\infty$, since then $-f^o$ and $-f^\wedge$ also take their values in $\mathbb{R}_+$.

The following calculus rules for the conjugates defined above may be useful; we consider only $f^\wedge$, but assertions similar to the ones in (a)-(c) are valid for f^o.

PROPOSITION 2.7.

(a) For any family $(f_i)_{i\in I}$ of functions on X one has $(\inf_{i\in I} f_i)^\wedge = \sup_{i\in I} f_i^\wedge$;

(b) for any function f on X and any $c \in \mathbb{R}$ one has $(f + c)^\wedge = f^\wedge - c$;

(c) for any function f on X and any $c \in \mathbb{R}_+$ one has $(cf)^\wedge = cf^\wedge$;

(d) if $A : X \to W$ is a continuous linear operator between two Banach spaces, if $g : W \to \mathbb{R}\cup\{\infty\}$ is a closed proper convex function such that $\mathbb{R}_+(A(X)+dom g)$ is a closed vector subspace of W, then

$$(g \circ A)^\wedge(x^*) = \inf\{g^\wedge(w^*) : w^* \in W^*,\ A^T(w^*) = x^*\}. \qquad \square$$

Proof. Only assertion (d) deserves a proof. We use the fact ([42] Theorem 2.2, [21] Proposition 4.11) that for $f := g \circ A$

$$f^\wedge(x^*) = \inf_{r\geqslant 0} \left(f^*(rx^*) - r\right),$$

with a similar formula for g and the classical formula

$$f^*(x^*) = \inf\{g^*(w^*) : w^* \in W^*,\ A^T(w^*) = x^*\},$$

valid under our assumptions ([2]), and we interchange the infima to get the announced formula. $\square$

Note that assertion (a) (resp. (b)) is valid for any duality (resp. conjugacy). Assertion (c) is not satisfied by all conjugacies; in particular it is not satisfied for the Fenchel-Moreau conjugacy.

3. Duality for reverse convex programs

In this section we consider the reverse convex program

$$(\mathcal{R})\quad \text{minimize } f(x)\ :\ x \in X\backslash C,$$

where C is an arbitrary subset of X, often taken to be convex. In [41] this problem is addressed in the case C is open, convex and contains 0 and the transformed problem

$$(\mathcal{T})\quad \text{maximize } f^\wedge(y)\ :\ y \in C^o$$

is introduced; in [50] p. 203 this problem is also studied in the case C is the interior of some convex subset D of X, so that $C^o = D^o$; in fact the conjugate which is used in [41, 50] is a function f^H which may differ at 0 from f^o. Here we introduce variants of $(\mathcal{T})$ which do not require openness of C. In the first one we use the strict polar set

$$C^\wedge := \{y \in Y : \ \forall x \in C \ \langle x, y\rangle < 1\}$$

of C introduced above. When C is open (or radiantly open in the sense that for each $x \in C$ there exists $t > 1$ such that $tx \in C$), this set coincides with the usual polar set C^o of C so that the following dual problem coincides with $(\mathcal{T})$:

$$(\mathcal{R}^\wedge) \quad \text{maximize } f^\wedge(y) \ : \ y \in C^\wedge.$$

The other two dual problems we introduce are

$$(\mathcal{R}^o) \quad \text{maximize } f^o(y) \ : \ y \in C^o.$$

$$(\mathcal{S}) \quad \text{maximize } f^o(y) \ : \ y \in C^\wedge.$$

In the following result we relate the values of these different dual problems. Moreover, we do not make use of the equivalence

$$x \in \mathrm{int} D \Leftrightarrow \sup\{\langle x, y\rangle : y \in D^o\} < 1$$

which is valid when D^o is compact but may fail in the general case, as observed in the introduction. It appears that the value of problem $(\mathcal{T})$ is not comparable to the value of problem $(\mathcal{R})$, unless f is a Thach function. In contrast, the values of problems $(\mathcal{R}^\wedge)$, $(\mathcal{R}^o)$ and $(\mathcal{S})$ are easily related to the value of problem $(\mathcal{R})$.

PROPOSITION 3.1.
(a) For any subset C of X and any function f one has

$$\begin{aligned} &\inf \mathcal{R} \leqslant -\sup \mathcal{R}^o, \quad \inf \mathcal{R} \leqslant -\sup \mathcal{R}^\wedge, \\ &\qquad -\sup \mathcal{T} \leqslant -\sup \mathcal{R}^o \leqslant -\sup \mathcal{S}, \\ &\qquad -\sup \mathcal{T} \leqslant -\sup \mathcal{R}^\wedge \leqslant -\sup \mathcal{S}. \end{aligned}$$

(b) In fact one has $\sup \mathcal{R}^o = \sup \mathcal{S}$.
(c) When f is a Thach function one has

$$\inf \mathcal{R} \leqslant -\sup \mathcal{T} = -\sup \mathcal{R}^o = -\sup \mathcal{S}.$$

(d) If $0 \in C$ and if C is evenly convex, in particular if C is open and convex, one has

$$\inf \mathcal{R} = -\sup \mathcal{R}^\wedge.$$

(e) If $0 \in C$ and if C is closed, convex, then one has

$$\inf \mathcal{R} = -\sup \mathcal{R}^\wedge = -\sup \mathcal{R}^o = -\sup \mathcal{S}.$$

Proof. (a) Since $X\backslash C$ contains the set $X\backslash C^{oo}$ of those $x \in X$ such that $\langle x, y\rangle > 1$ for some $y \in C^o$, one has

$$\begin{aligned}\inf\{f(x) : x \in X\backslash C\} &\leqslant \inf\{f(x) :\ x \in X\backslash C^{oo}\}\\ &\leqslant \inf\{f(x) :\ x \in X,\ y \in C^o,\ \langle x, y\rangle > 1\}\\ &\leqslant \inf\{-f^o(y) :\ y \in C^o\} = -\sup \mathcal{R}^o.\end{aligned}$$

Similar relations hold with $f^\wedge$ and $C^\wedge$.

The last two lines of (a) are immediate consequences of the relations $f^\wedge \geqslant f^o$, $C^\wedge \subset C^o$.

(b) Since f^o is l.s.c. and since for each $y \in C^o$ and each $t \in [0, 1)$ one has $ty \in C^\wedge$ we get

$$f^o(y) \leqslant \liminf_{t \nearrow 1} f^o(ty) \leqslant \sup \mathcal{S},$$

so that $\sup \mathcal{R}^o \leqslant \sup \mathcal{S}$.

(c) If f is a Thach function one has $f^\wedge = f^o$ and thus $\sup \mathcal{T} = \sup \mathcal{R}^o$.

(d) When C is evenly convex, by definition, for each $\overline{x} \in X\backslash C$ we can find some $y \in Y\backslash\{0\}$ and some $r \in \mathbb{R}$ such that

$$\langle \overline{x}, y\rangle \geqslant r > \langle x, y\rangle \ \ \forall x \in C.$$

As $0 \in C$ we have $r > 0$ and $\overline{y} := r^{-1}y \in C^\wedge$. Moreover, as $\langle \overline{x}, \overline{y}\rangle \geqslant 1$, i.e. $\overline{x} \in X\backslash G(\overline{y})$, we have

$$\sup \mathcal{R}^\wedge \geqslant f^\wedge(\overline{y}) = \sup\{-f(x) : x \in X\backslash G(\overline{y})\} \geqslant -f(\overline{x}).$$

Taking the supremum over $\overline{x} \in X\backslash C$ we get $\sup \mathcal{R}^\wedge \geqslant -\inf \mathcal{R}$.

(e) The proof is similar to the preceding one; moreover C is evenly convex, so that $\inf \mathcal{R} = -\sup \mathcal{R}^\wedge$. Given $\overline{x} \in X\backslash C$, the Hahn-Banach theorem yields some $y \in Y\backslash\{0\}$ and some $r \in \mathbb{R}$ such that

$$\langle \overline{x}, y\rangle > r > \langle x, y\rangle \ \ \forall x \in C.$$

Again, we have $r > 0$, $\overline{y} := r^{-1}y \in C^\wedge \subset C^o$ and $\langle \overline{x}, \overline{y}\rangle > 1$, hence $\overline{x} \in X\backslash H(\overline{y})$ and

$$\inf \mathcal{R} \leqslant -\sup \mathcal{R}^o \leqslant -\sup \mathcal{S} \leqslant -f^o(\overline{y}) = \inf\{f(x) : x \in X\backslash H(\overline{y})\} \leqslant f(\overline{x})$$

and the equalities follow by taking the infimum over $\overline{x} \in X\backslash C$. □

Let us present a characterization of optimal solutions. It is analogous to [42] Theorem 7.1.

PROPOSITION 3.2.

Suppose $0 \in C$ *and* C *is evenly convex (resp. closed and convex). For any* $\overline{x} \in X \backslash C$ *at which* f *is finite the following assertions are equivalent:*

(a) $\overline{x}$ *is a solution to* $(\mathcal{R})$;

(b) there exists $\overline{y} \in C^\wedge$ *(resp.* $\overline{y} \in C^o$*) such that* $\langle \overline{x}, \overline{y} \rangle \geqslant 1$ *(resp.* $\langle \overline{x}, \overline{y} \rangle > 1$*);*

(c) there exists an optimal solution $\overline{y}$ *of* $(\mathcal{R}^\wedge)$ *(resp.* $(\mathcal{R}^o)$ *) such that*

$$f(\overline{x}) + f^\wedge(\overline{y}) = 0, \quad \langle \overline{x}, \overline{y} \rangle \geqslant 1$$
$$\textit{(resp.} \;\; f(\overline{x}) + f^o(\overline{y}) = 0, \quad \langle \overline{x}, \overline{y} \rangle > 1\textit{).}$$

(d) there exists $\overline{y} \in C^\wedge$ *(resp.* $\overline{y} \in C^o$*) such that* $\overline{x}$ *is a minimizer of* f *on the half space* $\{x : \langle x, \overline{y} \rangle \geqslant 1\}$ *(resp.* $\{x : \langle x, \overline{y} \rangle > 1\}$*).*

(e) there exists $\overline{y} \in C^\wedge$ *(resp.* $\overline{y} \in C^o$*) such that* $f(\overline{x}) + f^\wedge(\overline{y}) = 0$ *(resp.* $f(\overline{x}) + f^o(\overline{y}) = 0$*).*

Moreover, any $\overline{y}$ *satisfying the conditions of (b) satisfies the conditions of (c). Furthermore one can take* $\overline{y} \in N^\wedge(C, \overline{x}) := \{y \in Y : \forall x \in C \; \langle x - \overline{x}, y \rangle < 0\}$.

Proof. The implications (a)⇒(b)⇒(c) follow from the proof above, since when $\overline{x}$ is an optimal solution to $(\mathcal{R})$, with $\overline{y}$ as in the proof of assertion (d), i.e. $\overline{y} \in C^\wedge$ and $\langle \overline{x}, \overline{y} \rangle \geqslant 1$, one has

$$\sup \mathcal{R}^\wedge \geqslant f^\wedge(\overline{y}) \geqslant -f(\overline{x}) = -\inf \mathcal{R} = \sup \mathcal{R}^\wedge$$

and a similar string of inequalities with $(\mathcal{R}^o)$ and f^o instead of $(\mathcal{R}^\wedge)$ and $f^\wedge$ when $\overline{y}$ is as in the proof of assertion (e) of the preceding proposition. The implications (c)⇒(d)⇒(e) are obvious.

When condition (e) holds, $\overline{x}$ and $\overline{y}$ are feasible for $(\mathcal{R})$ and $(\mathcal{R}^\wedge)$ respectively and

$$f(\overline{x}) = -f^\wedge(\overline{y}) = -\sup \mathcal{R}^\wedge = \inf \mathcal{R},$$

with similar relations with $(\mathcal{R}^o)$ and f^o instead of $(\mathcal{R}^\wedge)$ and $f^\wedge$, so that $\overline{x}$ is an optimal solution to $(\mathcal{R})$. The last assertion stems from the strict separation property of the preceding proof:

$$\langle x, \overline{y} \rangle < \langle \overline{x}, \overline{y} \rangle \quad \forall x \in C. \qquad \square$$

The preceding result can be interpreted in terms of subdifferentials. Let us recall that given a coupling function c, the c-subdifferential of f at $\overline{x} \in \operatorname{dom} f$ is the set $\partial^c f(\overline{x})$ of $\overline{y}$ such that $c(\overline{x}, \overline{y})$ is finite and

$$f(x) \geqslant f(\overline{x}) + c(x, \overline{y}) - c(\overline{x}, \overline{y}) \quad \forall x \in X.$$

Taking for c the couplings c^o, $c^\wedge$ and using the associated subdifferentials, ∂^o, $\partial^\wedge$ we see that $\overline{y} \in \partial^o f(\overline{x})$ (resp. $\overline{y} \in \partial^\wedge f(\overline{x})$) iff $\overline{x}$ is a minimizer of f over the half space $\{x : \langle x, \overline{y} \rangle > 1\}$ (resp. $\{x : \langle x, \overline{y} \rangle \geqslant 1\}$) and we get the following criteria.

COROLLARY 3.3. *Suppose* $0 \in C$ *and* C *is evenly convex (resp. closed and convex). If* $\overline{x} \in X\backslash C$ *is a solution to* $(\mathcal{R})$ *and if* $f(\overline{x})$ *is finite, then there exists* $\overline{y} \in \partial^{\wedge} f(\overline{x}) \cap N^{\wedge}(C, \overline{x})$ *(resp.* $\overline{y} \in \partial^{o} f(\overline{x}) \cap N^{\wedge}(C, \overline{x})$*).*

Proof. The result is a consequence of the preceding proposition and of the following well known characterization of $\partial^c f(\overline{x})$: $\overline{y} \in \partial^c f(\overline{x})$ iff $c(\overline{x}, \overline{y})$ is finite and

$$f(\overline{x}) + f^c(\overline{y}) = c(\overline{x}, \overline{y}).$$

In the present case this relation is equivalent to $f(\overline{x}) + f^c(\overline{y}) = 0$. □

Note that this last condition is necessary, but not sufficient. Such a situation prevails for nonconvex problems. The fact that the conditions of Proposition 3.2 are necessary and sufficient is in sharp contrast with such a general situation. Also, note that if one has an optimal solution $\overline{y}$ of $(\mathcal{R}^o)$ at one's disposal, then one gets a means to measure the accuracy of an approximate solution w to $(\mathcal{R})$: if $f(w) + f^o(\overline{y}) \leqslant \varepsilon$, then w is an ε-approximate solution to $(\mathcal{R})$.

4. Duality for quasiconvex maximization problems

In this section we consider the maximization problem

$$(\mathcal{M}) \quad \text{maximize } f(x) : \ x \in F$$

where $f : X \to \overline{I\!R}$ is an arbitrary function and the feasible set F is an arbitrary subset of X. We will impose generalized convexity assumptions on f and F to get sharp duality results. In [40–42] F is supposed to be a compact subset of X and the dual problem

$$(\mathcal{N}) \quad \text{minimize } f^{\wedge}(y) : \ y \in Y\backslash \text{int} F^o$$

is associated to $(\mathcal{M})$. Here we do not impose compactness assumptions and we introduce the dual problems

$$(\mathcal{M}^{\wedge}) \quad \text{minimize } f^{\wedge}(y) : \ y \in Y\backslash F^{\wedge}$$

$$(\mathcal{M}^o) \quad \text{minimize } f^o(y) : \ y \in Y\backslash F^o$$

and

$$(\mathcal{P}) \quad \text{minimize } f^o(y) : \ y \in Y\backslash F^{\wedge},$$

$$(\mathcal{Q}) \quad \text{minimize } f^{\wedge}(y) : \ y \in Y\backslash F^o.$$

When F is weakly compact $(\mathcal{M}^{\wedge})$ coincides with $(\mathcal{N})$, but this coincidence happens in other cases too (see Example 6.2 below in which F is a ball). The following proposition describes some other relationships between these problems.

PROPOSITION 4.1. *(a) For any function f and any feasible set F the following relations hold:*

$$\begin{aligned}
\sup \mathcal{M} &\geqslant -\inf \mathcal{M}^{\wedge} = \sup f^{\wedge\wedge}(F),\\
\sup \mathcal{M} &\geqslant -\inf \mathcal{M}^{o} = \sup f^{oo}(F),\\
\inf \mathcal{N} &\leqslant \inf \mathcal{M}^{\wedge} \leqslant \inf \mathcal{Q},\\
\inf \mathcal{P} &\leqslant \inf \mathcal{M}^{o} \leqslant \inf \mathcal{Q}.
\end{aligned}$$

(b) If f is evenly quasiconvex, $f(0) = \inf f(X)$ and $F \neq \{0\}$, then $\sup \mathcal{M} = -\inf \mathcal{M}^{\wedge}$.

(c) If f is l.s.c., quasiconvex, $f(0) = \inf f(X)$ and $F \neq \{0\}$, then $\sup \mathcal{M} = -\inf \mathcal{M}^{o}$.

(d) If f is a Thach function, then $\inf \mathcal{N} \leqslant \inf \mathcal{P} = \inf \mathcal{M}^{\wedge} \leqslant \inf \mathcal{Q} = \inf \mathcal{M}^{o}$.

(e) If X is the dual of the n.v.s. Y, if f is weakly l.s.c. and quasi-coercive on X, then* $\inf \mathcal{N} = \inf \mathcal{M}^{\wedge} = \inf \mathcal{Q}$.

Proof. As for $y \in F^o$ the set $\{x \in F : \langle x, y\rangle > 1\}$ is empty while it is nonempty when $y \in Y \backslash F^o$, one has

$$\begin{aligned}
\sup f^{oo}(F) &= \sup\{-f^o(y) : y \in Y,\ \langle x, y\rangle > 1,\ x \in F\}\\
&= \sup\{-f^o(y) : y \in Y\backslash F^o\} = -\inf \mathcal{M}^o.
\end{aligned}$$

This together with $f \geqslant f^{oo}$ implies the relations on the second line of the display in (a). The first line is similar. The third and the fourth lines are consequences of the inclusions $\mathrm{int} F^o \subset F^{\wedge} \subset F^o$ and of the inequality $f^{\wedge} \geqslant f^o$. Assertions (b) and (c) follow immediately from the first two lines of (a) and from Corollary 2.4. Assertion (d) is an immediate consequence of the definitions. Under the assumptions of assertion (e) $f^{\wedge}$ is u.s.c. Since for each $y \in Y\backslash \mathrm{int} F^o$ there exists a sequence (y_n) in $Y\backslash F^o$ converging to y, we have $f^{\wedge}(y) \geqslant \limsup_n f^{\wedge}(y_n) \geqslant \inf f^{\wedge}(Y\backslash F^o)$. Therefore $\inf \mathcal{N} \geqslant \inf \mathcal{Q}$; using the third line of assertion (a) we get the result. □

We observe that it may happen that $(\mathcal{N})$ is an unconstrained problem, so that the role of F vanishes, while $(\mathcal{M}^o)$ and $(\mathcal{M}^{\wedge})$ are still constrained problems. Such a situation appears when $X = \mathbb{R}^2$, $F = \mathbb{R} \times \mathbb{P}$, with $\mathbb{P} := (0, \infty)$, so that $F^o = \{0\} \times (-\infty, 0] = F^{\wedge}$, int $F^o = \emptyset$.

The solutions of the dual problem $(\mathcal{M}^{\wedge})$ can serve to locate the solutions of the primal problem, as in [40] which deals with the dual problem $(\mathcal{N})$. A similar result holds for the dual problem $(\mathcal{M}^o)$.

PROPOSITION 4.2.

(a) If $\overline{y}$ is a solution to $(\mathcal{M}^{\wedge})$, then any $\overline{x} \in F$ such that $\langle \overline{x}, \overline{y}\rangle \geqslant 1$ (and there exist such $\overline{x}$'s) is a maximizer of $f^{\wedge\wedge}$ on F, hence is a solution to $(\mathcal{M})$ when $f^{\wedge\wedge} = f$.

(b) If $\overline{x}$ is a maximizer of $f^{\wedge\wedge}$ on F, then any minimizer of $f^{\wedge}$ on the half-space $G'(\overline{x}) := \{y \in Y : \langle \overline{x}, y\rangle \geqslant 1\}$ is a solution to $(\mathcal{M}^{\wedge})$.

Proof. (a) If $\overline{y}$ belongs to the set of solutions of $(\mathcal{M}^\wedge)$, one has $\overline{y} \in Y\backslash F^\wedge$, so that there exists at least one $\overline{x} \in F$ such that $\langle \overline{x}, \overline{y}\rangle \geqslant 1$. For such a $\overline{x}$ one has

$$f^{\wedge\wedge}(\overline{x}) \geqslant -f^\wedge(\overline{y}) = -\inf \mathcal{M}^\wedge = \sup f^{\wedge\wedge}(F),$$

so that $\overline{x}$ is a maximizer of $f^{\wedge\wedge}$ on F.

(b) For each maximizer $\overline{x}$ of $f^{\wedge\wedge}$ on F, the set $G'(\overline{x})$ is contained in the feasible set $Y\backslash F^\wedge$ of $(\mathcal{M}^\wedge)$. Thus, if $\overline{y}$ is a minimizer of $f^\wedge$ on $G'(\overline{x})$, one has

$$-\inf \mathcal{M}^\wedge = \sup f^{\wedge\wedge}(F) = f^{\wedge\wedge}(\overline{x}) = -\inf_{y\in G'(\overline{x})} f^\wedge(y) = -f^\wedge(\overline{y}) \leqslant -\inf \mathcal{M}^\wedge,$$

so that $\overline{y}$ is a solution to $(\mathcal{M}^\wedge)$. □

5. Combination of both duality results

In this section we combine the results of the preceding two sections. Observing that the dual problems $(\mathcal{M}^\wedge)$ and $(\mathcal{M}^o)$ of the preceding section are reverse convex programs of the type studied in Section 3, we can consider their dual problems for $(\mathcal{R}) = (\mathcal{M}^\wedge)$ or $(\mathcal{R}) = (\mathcal{M}^o)$:

$$\begin{array}{ll} (\mathcal{M}^{\wedge\wedge}) & \text{maximize } f^{\wedge\wedge}(x) : \ x \in F^{\wedge\wedge} \\ (\mathcal{M}^{oo}) & \text{maximize } f^{oo}(x) : \ x \in F^{oo}. \end{array}$$

The following result is an immediate consequence of Propositions 3.1 and 4.1, taking into account the facts that F^o (resp. $F^\wedge$) is closed convex (resp. is evenly convex) and contains 0 and that $F \subset F^{oo}$ (resp. $F \subset F^{\wedge\wedge}$).

PROPOSITION 5.1. *For any function f and any feasible set F the following relations hold*

$$\begin{aligned} \sup \mathcal{M} &\geqslant -\inf \mathcal{M}^o \ = \ \sup \mathcal{M}^{oo}, \\ \sup \mathcal{M} &\geqslant -\inf \mathcal{M}^\wedge \ = \ \sup \mathcal{M}^{\wedge\wedge}. \end{aligned}$$

If $f = f^{oo}$ (resp. $f = f^{\wedge\wedge}$) then the first (resp. second) inequality is an equality.

On the other hand, starting from problem $(\mathcal{R})$, we observe that problems $(\mathcal{R}^o)$ and $(\mathcal{R}^\wedge)$ are in the form of $(\mathcal{M})$. Therefore we can use their dual problems

$$\begin{array}{ll} (\mathcal{R}^{\wedge\wedge}) & \text{maximize } f^{\wedge\wedge}(x) \ : \ x \in X\backslash C^{\wedge\wedge} \\ (\mathcal{R}^{oo}) & \text{maximize } f^{oo}(x) \ : \ x \in X\backslash C^{oo}. \end{array}$$

PROPOSITION 5.2. *For any function f and any feasible set F the following relations hold*

$$\begin{aligned} \inf \mathcal{R} &\leqslant -\sup \mathcal{R}^o \ = \ \inf \mathcal{R}^{oo}, \\ \inf \mathcal{R} &\leqslant -\sup \mathcal{R}^\wedge \ = \ \inf \mathcal{R}^{\wedge\wedge}. \end{aligned}$$

If $C = C^{oo}$ (resp. $C = C^{\wedge\wedge}$) then the first (resp. second) inequality is an equality.

Proof. Here we use the fact that the objective of $(\mathcal{R}^o)$ is the function f^o which satisfies $(f^o)^{oo} = f^o$, and we apply Proposition 3.1 (a) and Proposition 4.1 (c) for the first assertion. The second assertion is a consequence of Proposition 3.1 (e). Similar arguments hold for $(\mathcal{R}^\wedge)$. □

6. Comparisons and applications

In [21] we deal with the connection between the preceding results and the Toland-Singer duality theory. A complete comparison with other existing duality relationships is out of the scope of the present paper. Let us show however on important examples how our results apply and can be related to existing ones.

EXAMPLE 6.1. Let us consider the maximization problem $(\mathcal{M})$ in which the feasible set F is a polyhedron in a finite dimensional space X given as the convex hull of a finite family $(a_i)_{i=1,\dots,n}$ of points of X. Then $(\mathcal{M}^\wedge)$ and $(\mathcal{M}^o)$ take the forms

$$(\mathcal{M}^\wedge) \ \ \text{minimize } f^\wedge(y) \ : \exists\, i \in \{1, \dots, n\}\ \langle y, a_i \rangle \geqslant 1$$

$$(\mathcal{M}^o) \ \ \text{minimize } f^o(y) \ : \exists\, i \in \{1, \dots, n\}\ \langle y, a_i \rangle > 1.$$

They can be solved by considering separately the problems of minimizing $f^\wedge$ (resp. f^o) on the n half-spaces $H_i := \{y : \langle y, a_i \rangle \geqslant 1\}$ (resp. $\text{int}H_i$). These problems are simply constrained minimization problems which can be treated with parallel algorithms. When f is a positive definite quadratic form, these problems are quadratic minimization problems with linear constraints.

EXAMPLE 6.2. Let X be a real Hilbert space with unit ball B_X, let $F := B_X$ and f be as in Example 2.1: $f(x) = \frac{1}{2}(Ax \mid x)$, with A injective, symmetric and semi-definite positive. Then $\text{int}F^o = F^\wedge$ and f is a Thach function so that the dual problems $(\mathcal{M}^\wedge)$, $(\mathcal{N})$, $(\mathcal{P})$ of the problem $(\mathcal{M})$ of maximizing f on F coincide, have the same value as the dual problems $(\mathcal{M}^o) = (\mathcal{Q})$ and are given by

$$(\mathcal{M}^\wedge) \ \ \text{minimize } -\frac{1}{2}\left\|A^{-\frac{1}{2}}y\right\|^{-2} \ \ \text{for } y \in R(A^{\frac{1}{2}})\backslash\text{int}B_X.$$

When the primal problem $(\mathcal{M})$ has a solution (in particular when A is a compact operator) it is an eigenvector corresponding to the largest eigenvalue α of A. In such a case, assertion (b) of Proposition 4.2 gives a means to solve $(\mathcal{M}^\wedge)$ and we find that $(\mathcal{M})$ has the value $\frac{1}{2}\alpha$ and $(\mathcal{M}^\wedge)$ has the value $-\frac{1}{2}\alpha$. In the simple case in which $R(A^{\frac{1}{2}}) = R(A)$, the value of $(\mathcal{M}^\wedge)$ coincides with the opposite of the value of $(\mathcal{M})$ as then $f^{\wedge\wedge} = f$.

Note that solving $(\mathcal{M}^\wedge)$ is equivalent to solving the reverse convex problem

$$(\mathcal{R}) \ \ \text{minimize } \|x\| \ \ \text{for } x \in X \text{ satisfying } \left\|A^{\frac{1}{2}}x\right\| \geqslant 1$$

for which there is no need to suppose A is injective. Note that the dual problem $(\mathcal{R}^o)$ coincides with the dual problem $(\mathcal{T})$ and is equivalent to the maximization of the function $y \mapsto \left\| A^{\frac{1}{2}} y \right\|$ under the constraint $\|y\| \leqslant 1$, hence is equivalent to the maximization of f under this constraint, our original problem.

EXAMPLE 6.3. The problem $(\mathcal{R})$ considered in the preceding example is a special case of the problem

$$(\mathcal{R}) \quad \text{minimize } f(x) \text{ for } x \in X \text{ satisfying } Ax \in W \backslash B,$$

where A is a continuous linear operator from X into another n.v.s. W and B is a convex subset of W containing 0. Since $A^{-1}(W \backslash B) = X \backslash A^{-1}(B)$, the dual problems $(\mathcal{T})$ and $(\mathcal{R}^o)$ of $(\mathcal{R})$ involve the feasible set C^o, where $C := A^{-1}(B)$. One always has $A^T(B^o) \subset C^o$ and when W and X are complete and the transversality condition

$$\mathbb{R}_+(A(X) + \overline{B}) = W \tag{6.2}$$

is satisfied, where $\overline{B}$ is the closure of B, one has $D^o = A^T(\overline{B}^o) = A^T(B^o)$ for $D := A^{-1}(\overline{B})$ (see [2]). It follows that $\inf \mathcal{R} \leqslant -\sup \mathcal{R}^o \leqslant -\sup f^o(A^T(B^o))$. As in [40] we note that the problem

$$(\mathcal{A}) \quad \text{maximize } f^o(A^T(z)) \text{ for } z \in B^o \tag{6.3}$$

may be much simpler than the dual problem $(\mathcal{R}^o)$. In particular, if the dimension n of X is large while the dimension m of W is small, the auxiliary problem $(\mathcal{A})$ is more tractable than $(\mathcal{R}^o)$. Although the value of $(\mathcal{A})$ provides only an estimate for the value of $(\mathcal{R})$, when the set B is polyhedral or when B is closed convex and condition (6.2) is satisfied with W, X complete, Proposition 3.1 (e) shows that $\inf \mathcal{R} = -\sup \mathcal{A}$.

EXAMPLE 6.4. Several authors have considered the case in which the constraint set C is defined by inequalities ([13–15], [40–50] ...) In Lemaire [13] the following problem is considered:

$$(\mathcal{L}) \quad \text{minimize } g(x) : \ x \in X, \ h(x) > 0,$$

where g, h are two extended real-valued proper convex functions. This problem is a special case of problem $(\mathcal{R})$ with $C := \{x \in X : h(x) \leqslant 0\}$, a closed convex subset of X. Conversely, taking $g = f$, $h = \iota_C$, we see that problem $(\mathcal{R})$ can be put under the form of problem $(\mathcal{L})$. However, the dual problem of [13] uses the classical convex conjugates g^* and h^* of g and h respectively. In its simplest form, assuming that $\operatorname{dom} g = X$ and $\inf h \leqslant 0$, it is as follows:

$$(\mathcal{L}^*) \quad \text{minimize } \sup_{t \geqslant 0}(th^*(y) - g^*(ty)) : \ y \in Y \backslash \{0\}, \ h^*(y) < \infty.$$

The relation

$$f^{\wedge}(y) = \inf_{t \geqslant 0}(f^*(ty) - t) \qquad \forall y \in Y \backslash \{0\}$$

proved in [42] Theorem 2.2, which holds when f is closed proper convex, can serve to relate problems $(\mathcal{L}^*)$ and $(\mathcal{R}^{\wedge})$. Let us relate problems $(\mathcal{L}^*)$, $(\mathcal{R}^{\wedge})$ and $(\mathcal{R}^{o})$ for the important problem of finding the greatest radius of an open ball with center 0 and contained in an open convex subset C of X containing 0. Then $(\mathcal{R})$ takes the form

$$(\mathcal{B}) \quad \text{minimize } \|x\| \quad : \; x \in X \backslash C.$$

Since for $f := \|\cdot\|$ one has $f^{\wedge} = f^{o} = -\|\cdot\|^{-1}$, with the convention $0^{-1} = \infty$, the value $\gamma := \sup \mathcal{R}^{\wedge}$ of $(\mathcal{R}^{\wedge})$ is

$$\gamma = \sup\{-\|y\|^{-1} : y \in C^{\wedge}\}.$$

Since C is open, $C^{\wedge} = C^{o}$ and γ is also the value of $(\mathcal{R}^{o})$. On the other hand, the support function $h_C := \iota_C^*$ of C being positively homogeneous and nonnegative, the value β^* of the dual problem of $(\mathcal{B})$ in the sense of [13] is easily seen to be

$$\beta^* = \inf\{\iota_C^*(v) : v \in Y, \|v\| = 1\},$$

or, equivalently,

$$\beta^* = \inf\{\iota_C^*(y) : y \in Y, \|y\| \geqslant 1\}.$$

Now for each $y \in C^{\wedge} = C^{o}$, $y \neq 0$ we have $h_C(y) \leqslant 1$, so that, for $v := \|y\|^{-1}y$ we have $\|y\|^{-1} \geqslant \|y\|^{-1}h_C(y) = h_C(v) \geqslant \beta^*$. Taking the infimum on $C^{\wedge}$ we get $-\gamma \geqslant \beta^*$. Now, given $r > \beta^*$ we can find $v \in Y$ such that $\|v\| = 1$, $h_C(v) < r$. Then, as $0 \in C$, we have $r > 0$, $y := r^{-1}v \in C^{\wedge}$ and $\|y\|^{-1} = r$, so that $-\gamma \leqslant r$. Therefore $-\gamma = \beta^*$. We obtain that the value β of problem $(\mathcal{B})$ can be expressed in two other different ways. Moreover, the estimate $\beta \leqslant -\gamma$ of Proposition 3.1 (a) does not assume that C is convex.

EXAMPLE 6.5. (Burkard, Oettli and Thach [4, 41, 43] for the case $m = 3$) Let $a_1, \ldots, a_n$ be a family of vectors of the m dimensional Euclidean space $\mathbb{R}^m$ and let w_j be a weight associated with each vector a_j for $j = 1, \ldots, n$. The generalized knapsack problem we consider consists in choosing a subfamily $(a_{j_1}, \ldots, a_{j_k})$ of vectors such that $a_{j_1} + \ldots + a_{j_k}$ has a maximum length and the sum $w_{j_1} + \ldots + w_{j_k}$ of the corresponding weights does not exceed 1. When the vectors a_j are colinear to a given vector, this problem reduces to an ordinary knapsack problem. Introducing $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$ we can formulate this problem as

$$(\mathcal{K}) \quad \text{maximize } \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_j a_{i,j}\Big)^2 : \; x \in F,$$

where $a_{i,j}$ is the ith component of a_j and the feasible set F is the discrete set

$$F := \left\{ x = (x_1, \dots, x_n) \in \{0,1\}^n : \sum_{i=1}^{n} w_i x_i \leqslant 1 \right\}.$$

This set contains 0 and is compact, so that $F^\wedge = \mathrm{int} F^o$. Moreover, the objective function f of $(\mathcal{K})$ is a continuous convex quadratic function, so that $f^\wedge = f^o$. Therefore the difficulty in choosing the appropriate dual problem for $(\mathcal{M}) = (\mathcal{K})$ among those we introduced is reduced: we have $(\mathcal{N}) = (\mathcal{M}^\wedge) = (\mathcal{P})$ and $(\mathcal{M}^o) = (\mathcal{Q})$. Moreover, if we modify the value of f at zero in setting $f(0) = -\infty$, we have $f^{oo} = f$. Inasmuch $\inf_i w_i \leqslant 1$, F is not reduced to $\{0\}$; moreover $f(0) = \inf f(X)$. Thus Proposition 4.1 (c) shows that $\sup \mathcal{K} = -\inf \mathcal{M}^o$. Since $f^\wedge = f^o$ and since $F^\wedge \subset F^o$, we get $-\inf \mathcal{M}^\wedge \geqslant -\inf \mathcal{M}^o$, hence

$$\sup \mathcal{K} = -\inf \mathcal{M}^o = -\inf \mathcal{M}^\wedge.$$

Introducing the operator $A : \mathbb{R}^m \to \mathbb{R}^n$ given by $A(x) := (a_1.x, \dots, a_n.x)^T$, where $a_i.x$ denotes the scalar product in $\mathbb{R}^m$, and using Proposition 2.7, we see that $f(x) = \|Ax\|^2$, hence

$$f^o(y) = f^\wedge(y) = \inf \left\{ -\frac{1}{\|z\|^2} : z \in \mathbb{R}^m,\ A^T(z) = y \right\},$$

the conjugate $\left(\|\cdot\|^2\right)^\wedge = \left(\|\cdot\|^2\right)^o$ of the square of the norm being $-\|\cdot\|^{-2}$. As in Example 6.3 we are led to make a change of variable in problem $(\mathcal{M}^\wedge)$ and to consider the equivalent problem

$$(\mathcal{K}^\wedge) \quad \text{minimize } \|z\|^2 : \ A^T(z) \in Y \backslash F^\wedge$$

in the sense that a solution $\overline{z}$ to $(\mathcal{K}^\wedge)$ yields a solution $\overline{y} = A^T(\overline{z})$ to $(\mathcal{M}^\wedge)$. Now we observe that $F^\wedge$ is a finite intersection of open half-spaces. Therefore solving $(\mathcal{K}^\wedge)$ amounts to finding the point of the boundary of a convex polyhedron which is closest to the origin.

We may conclude from the preceding examples that the abundance of the dual problems we exhibited is an advantage rather than an obstacle, for it allows to use various properties which may help to solve the original problem.

It has been pointed out to the author by M. Volle (personal communication) that the conjugacy of positive functions introduced by Rubinov and Simsek [30, 31] can be deduced from the preceding conjugacy by taking logarithms. More precisely, the conjugate of a positive function q according to [30] is

$$q^{RS}(y) := \sup \left\{ q(x)^{-1} : \langle x, y \rangle > 1 \right\},$$

so that

$$\log q^{RS}(y) = \sup \{ -\log q(x) : \langle x, y \rangle > 1 \} = (\log \circ q)^o(y).$$

A number of results from [30] could be derived from [52] or from the results of the present paper by taking the preceding observation into account. However, the anticonvex problems studied here are not considered in [30, 31]. Let us also add that a rich class of problems involving functions which are convex along rays or quasiconvex along rays, but not necessarily convex or quasiconvex functions are considered by Prof. Rubinov and his co-authors and these problems are out of the scope of the present paper.

Added in proof. It has been pointed out by Prof. Rubinov that Example 6.2 above is related to Proposition 4.2 and Examples 4.1 and 4.2 of the paper: A. Rubinov and B. Glover, Toland-Singer formula cannot distinguish a global minimizer from a choice of stationary points, Numer. Funct. Anal. Optim. 20 (1999), 99–120.

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